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Turning the quantum group invariant XXZ spin-chain Hermitian: a conjecture on the invariant product

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Abstract
This is a continuation of a previous joint work with Robert Weston on the quantum group invariant *XXZ* spin-chain (*Preprint* math-ph/0703085). The previous results on quasi-Hermiticity of this integrable model are briefly reviewed and then connected with a new construction of an inner product with respect to which the Hamiltonian and the representation of the Temperley–Lieb algebra become Hermitian. The approach is purely algebraic, one starts with the definition of a positive functional over the Temperley–Lieb algebra whose values can be computed graphically. Employing the Gel'fand–Naimark–Segal (GNS) construction for C*-algebras a self-adjoint representation of the Temperley–Lieb algebra is constructed when the deformation parameter q lies in a special section of the unit circle. The main conjecture of this paper is the unitary equivalence of this GNS representation with the representation obtained in the previous paper employing the ideas of PT-symmetry and quasi-Hermiticity. An explicit example is presented.

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1. Introduction

This is a continuation of a previous paper [1] where the Hermiticity properties of the quantum group invariant *XXZ* spin-chain Hamiltonian [2, 3]

$$H = \frac{1}{2}\sum_{n=1}^{N-1}\left\{\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta_+\left(\sigma_n^z\sigma_{n+1}^z - 1\right)\right\} + \Delta_-\frac{\sigma_1^z - \sigma_N^z}{2} \tag{1}$$

have been investigated. Here the anisotropy parameters $\Delta_\pm$ are defined in terms of a single variable q,

$$\Delta_\pm = \frac{q \pm q^{-1}}{2}, \tag{2}$$

and $\{\sigma_n^{x,y,z}\}$ denote the Pauli matrices acting on the nth site of the spin-chain represented by the state space $\mathfrak{H} = V^{\otimes N}$ with V being isomorphic to $\mathbb{C}^2$. The above Hamiltonian, besides belonging to an integrable model, is distinguished by its $U_q(sl_2)$-invariance. Namely, one has the following representation $U_q(sl_2) \to \mathrm{End}\,\mathfrak{H}$ in terms of the matrices

$$q^{\pm S^z} = \prod_{n=1}^{N} q^{\pm\sigma_n^z/2}, \qquad S^{\pm} = \sum_{n=1}^{N} q^{\frac{\sigma^z}{2}} \otimes \cdots \otimes \underset{n^{\text{th}}}{\sigma^{\pm}} \otimes q^{-\frac{\sigma^z}{2}} \cdots \otimes q^{-\frac{\sigma^z}{2}}, \tag{3}$$

which obey the familiar $U_q(sl_2)$-commutation relations

$$q^{S^z} q^{-S^z} = q^{-S^z} q^{S^z} = 1, \qquad q^{S^z} S^{\pm} q^{-S^z} = q^{\pm 1} S^{\pm}, \qquad [S^+, S^-] = [2S^z]_q. \tag{4}$$

Here, as usual, we have set

$$[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}.$$

The case when the deformation parameter q lies on the unit circle $\mathbb{S}^1$ is of particular interest, since then the corresponding lattice model is believed to correspond in the thermodynamic limit to a CFT with central charge [2, 3]

$$c = 1 - \frac{6}{(r-1)r}, \qquad q = \exp\left(\frac{\mathrm{i}\pi}{r}\right), \qquad r \in \mathbb{R}. \tag{5}$$

However, for these values of q one easily verifies that H is non-Hermitian with respect to the conventional scalar product,

$$q \in \mathbb{S}^1 : \qquad H \neq H^* \tag{6}$$

with $*$ denoting the Hermitian adjoint (or conjugate transpose in the case of matrices). It is therefore not clear whether (1) constitutes a well-defined quantum system or is even diagonalizable. Numerical computations show, however, that the Hamiltonian possesses real spectrum for *all* $q \in \mathbb{S}^1$. This raises the questions whether for all values on the unit circle the Hamiltonian (1) can be related to a Hermitian Hamiltonian via introducing a different inner product or performing a similarity transformation. More precisely, one can ask if for each $q \in \mathbb{S}^1$ there exists a *positive, Hermitian and invertible* operator $\eta : \mathfrak{H} \to \mathfrak{H}$ such that

$$\eta H = H^* \eta. \tag{7}$$

The existence of such a map η would enable one to introduce a new inner product

$$\langle \cdot, \cdot \rangle_\eta : \mathfrak{H} \times \mathfrak{H} \to \mathbb{C}, \qquad \langle x, y \rangle_\eta := \langle x, \eta y \rangle \tag{8}$$

with respect to which the Hamilton operator becomes Hermitian,

$$\langle x, Hy \rangle_\eta = \langle Hx, y \rangle_\eta, \qquad x, y \in \mathfrak{H}. \tag{9}$$

The Hamiltonian is then called quasi-Hermitian [4, 5] and in this paper we shall use this terminology together with closely related concepts from PT-symmetry (parity and time reversal), see e.g. [6, 7] for recent reviews.

Parametrize the deformation parameter q as in (5). Then for the two special cases

(i) $r = 3, 4, 5, \ldots, N$ and
(ii) $r \in (N, \infty) \subset \mathbb{R}$ (here N is the number of sites)

one can argue on purely algebraic and representation theoretic grounds that a map η as detailed above must exist, albeit in case (i) one needs to carry out a reduction of the state space first. That is, one needs to replace $\mathfrak{H} \to \mathfrak{H}_{\text{red}}$ with $\mathfrak{H}_{\text{red}} \subset \mathfrak{H}$ being a proper subspace. In what follows this replacement for (i) shall always be implicitly understood.

The mentioned algebraic framework rests on the identification of the Hamiltonian (1) as an element in the Temperley–Lieb algebra [8, 9] $TL_N(q)$ which is obtained from $N-1$ generators $\{e_1, e_2, \ldots, e_{N-1}\}$ satisfying the commutation relations

$$\begin{aligned} &e_i^2 = -(q+q^{-1})e_i, \\ &e_i e_{i\pm 1} e_i = e_i, \\ &e_i e_j = e_j e_i, \qquad |i-j|>1. \end{aligned} \tag{10}$$

There is a representation $\pi_{TL} : TL_N(q) \to \mathrm{End}\, V^{\otimes N}$ in terms of Pauli matrices by mapping $e_i \mapsto \pi_{TL}(e_i) = E_i$, where the E_i's are defined in terms of the local Hamiltonians:

$$H = \sum_{i=1}^{N-1} E_i, \qquad E_i = \frac{\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y}{2} + \Delta_+ \frac{\sigma_i^z\sigma_{i+1}^z - 1}{2} + \Delta_- \frac{\sigma_i^z - \sigma_{i+1}^z}{2}. \tag{11}$$

For cases (i) and (ii) the existence of self-adjoint representations of the Temperley–Lieb algebra is known (see e.g. [10, 11, 13] and references therein) and one can then argue using algebraic techniques that these representations must be similar to (11). From the existence of the similarity transformation with respect to the representations of $TL_N(q)$ one can obviously deduce that η exists and that it must satisfy the more stringent condition

$$\eta E_i = E_i^* \eta. \tag{12}$$

There are, however, drawbacks to this approach. For values of $q = \exp(\mathrm{i}\pi/r) \in \mathbb{S}^1$ other than (i) or (ii) the mentioned algebraic line of argument breaks down. In fact, the requirement (12) proves to be too restrictive in general. Moreover, besides an abstract proof of existence one wishes for practical purposes to have an explicit construction of the map η.

In [1] such an explicit construction has been carried out drawing on earlier results known as 'quantum group reduction' [14] in order to make the replacement $\mathfrak{H} \to \mathfrak{H}_{\mathrm{red}}$ for $r = 3, 4, 5, \ldots, N$. This replacement removes non-trivial Jordan blocks of the Hamiltonian. In both cases, (i) and (ii), we found explicit algebraic expressions for η utilizing concepts related to the quantum analogue of Schur–Weyl duality [15]. The results in [1] tie the abstract representation theoretic approach discussed above to the notions of quasi-Hermiticity and PT-symmetry and provide one of the few examples of a non-Hermitian Hamiltonian where quasi-Hermiticity has been proved by an *explicit* and *exact* construction of the operator η.

The new result in the present paper is an entirely independent approach to introduce the new inner product (8) and thus to implicitly define η for case (ii), i.e. when $r \in (N, \infty)$. Namely, we will define for each fixed spin sector $\mathfrak{H}_n$ (here n is the number of down spins) a positive functional

$$\omega_n : TL_N(q) \to \mathbb{R}, \qquad n = 0, 1, 2, \ldots, N$$

over the Temperley–Lieb algebra using only its graphical representation in terms of Kauffman diagrams [12]. We then employ a well-known tool from the representation theory of C^*-algebras, the Gel'fand–Naimark–Segal (GNS) construction, which provides us with an inner product $\langle\cdot,\cdot\rangle_{\omega_n}$ in terms of the functional ω_n and an associated Hilbert space $\mathfrak{H}_n^{\mathrm{GNS}}$. The GNS construction is such that the Temperley–Lieb algebra generators e_i act as self-adjoint operators over $\mathfrak{H}_n^{\mathrm{GNS}}$, thus giving rise to a Hermitian Hamiltonian $h = \sum_{i=1}^{N-1} e_i$.

The main conjecture of this paper is that the two self-adjoint representations, one induced by the map η constructed in [1] and the other by the functionals $\{\omega_n\}_{n=0}^N$, are unitarily equivalent. That is, there exists for each $n = 0, 1, \ldots, N$ an isomorphism $U_n : \mathfrak{H}_n^{\mathrm{GNS}} \to \mathfrak{H}_n$ satisfying

$$U_n h U_n^{-1} = H, \qquad h = \sum_{i=1}^{N-1} e_i \tag{13}$$

Table 1. Transformations under parity, time and spin reversal.

Operator	Temperley–Lieb	Quantum group
Parity reversal	$PE_kP = E^*_{N-k}$	$PS^{\pm}P = (S^{\mp})^*$
Time reversal	$TE_kT = E^*_k$	$TS^{\pm}T = (S^{\mp})^*$
Spin reversal	$RE_kR = E^*_k$	$RS^{\pm}R = (S^{\pm})^*$

and the identity

$$\langle v, w\rangle_{\omega_n} = \langle U_n v, U_n w\rangle_\eta, \qquad v, w \in \mathfrak{H}_n^{\mathrm{GNS}}. \tag{14}$$

The left-hand side of the last equality can be computed by purely graphical means. This provides us with a novel, efficient formalism to investigate the spectrum, eigenvectors and, in the long term, correlation functions of the quantum group invariant *XXZ* Hamiltonian. Moreover, the presented graphical definition of the self-adjoint representation is also a new representation theoretic result for the Temperley–Lieb algebra which (to the best of the author's knowledge) appears not to be contained in the extensive literature on this subject. The extension of this graphical calculus to the root of unity case, where a reduction of the state space needs to be carried out first, is still an open problem.

2. Quasi-Hermiticity of the XXZ chain

In order that the reader can fully appreciate the main conjecture of this paper we briefly review the results obtained for the quasi-Hermiticity operator η in the previous paper [1].

2.1. Discrete symmetries of the Hamiltonian

We recall the following definitions from [1].

Definition 1 (parity, time and spin-reversal). *Let $V = \mathbb{C}v_{+1/2} \oplus \mathbb{C}v_{-1/2}$ then we define the linear operator P on $\mathfrak{H} = V^{\otimes N}$ by setting*

$$Pv_{\alpha_1} \otimes v_{\alpha_2} \cdots \otimes v_{\alpha_N} = v_{\alpha_N} \otimes v_{\alpha_{N-1}} \cdots \otimes v_{\alpha_1}, \qquad \alpha_i = \pm 1/2. \tag{15}$$

In contrast, the operator T acts on the basis vectors as the identity,

$$Tv_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N} = v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N}, \tag{16}$$

but is defined to be antilinear, such that

$$T\lambda v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N} = \bar{\lambda} v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N}, \qquad \lambda \in \mathbb{C}. \tag{17}$$

Thus, any matrix A (such as the Hamiltonian $A = H$) is transformed into its complex conjugate under the adjoint action of T, $TAT = \bar{A}$. Finally, we introduce the (linear) spin-reversal operator R by setting

$$Rv_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N} = v_{-\alpha_1} \otimes \cdots \otimes v_{-\alpha_N}.$$

One now easily computes the transformation properties of the quantum group and Temperley–Lieb generators under the adjoint action of the involutions P, T, R. They are summarized in table 1.

As a trivial consequence of the relation in table 1, we have the vanishing commutators

$$[PT, H] = [PR, H] = [RT, H] = 0 \tag{18}$$

for the Hamiltonian H. Since T is antilinear and neither P nor R are positive operators, none of these relations is sufficient to prove quasi-Hermiticity. Nevertheless, these discrete transformations play an essential role in the formulation of the quasi-Hermiticity operator η.

2.2. Three expressions for the quasi-Hermiticity operator η

We have the following three equivalent expressions for the quasi-Hermiticity operator η. For their derivation and proofs we refer the reader to [1].

Expression 1: η as sum of projectors. The first expression for the quasi-Hermiticity operator η is closely related to the idea of quantum group reduction. First one introduces a special basis, called the path basis, which decomposes the state space with respect to the action of the quantum group. Broadly speaking one successively 'fuses' the spin-1/2 modules $V = V_1$ at each lattice site to (higher) spin-j modules V_{2j}.

Fact. The finite-dimensional irreducible representations of $U_q(sl_2)$ are isomorphic to the following modules $V_{2j} \cong \mathbb{C}^{2j+1}$ indexed by $j \in \frac{1}{2}\mathbb{N}$ and defined through the maps $\pi_j : U_q(sl_2) \to \operatorname{End} V_{2j}$ with

$$\begin{aligned} &\pi_j(s^{\pm})|j,m\rangle = \sqrt{[j \mp m]_q[j \pm m + 1]_q}\,|j, m \pm 1\rangle, \\ &\pi_j(q^{s^z})|j,m\rangle = q^m|j,m\rangle, \qquad m = -j, -j+1, \ldots, j-1, j. \end{aligned} \tag{19}$$

Clearly, there is a choice in which order to execute this 'fusing procedure' and this choice is encoded in a path $\boldsymbol{j} = (j_0 = 0, j_1 = 1/2, j_2, \ldots, j_N)$. For instance, if $N = 3$ we can fuse the first two sites to a spin-1 or a spin-0 module and then in the next step obtain either a spin-3/2 or spin-1/2 module yielding the three paths

$$\boldsymbol{j} = (0, 1/2, 1, 3/2), (0, 1/2, 1, 1/2), (0, 1/2, 0, 1/2).$$

Here we have rooted each path at $j_0 = 0$ as it is convention in the literature. The corresponding basis vectors are given explicitly by the following formulae.

Let $\boldsymbol{j} = (j_0, j_1, j_2, \ldots, j_N)$ be a path on the sl_2-Bratelli diagram, i.e. the set of sequences specified as follows:

$$\Gamma = \{\boldsymbol{j} = (j_0, j_1, j_2 \ldots, j_N) | j_0 = 0, j_k \geqslant 0, j_{k+1} = j_k \pm 1/2\}. \tag{20}$$

Then we define for each $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_i = \pm 1/2$ the vectors

$$|\boldsymbol{j}, m\rangle = \sum_{|\alpha|=m} \langle \alpha|\boldsymbol{j}, m\rangle\, v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N}, \qquad m = -j_N, -j_N + 1, \ldots, 0, \ldots, j_N \tag{21}$$

with

$$\langle \alpha|\boldsymbol{j}, m\rangle = \prod_{k=1}^{N-1} \left| \begin{matrix} j_k & \frac{1}{2} & j_{k+1} \\ \sum_{i \leqslant k} \alpha_i & \alpha_{k+1} & \sum_{i \leqslant k+1} \alpha_i \end{matrix} \right|_q, \qquad m = |\alpha| = \sum_{k=1}^{N} \alpha_k. \tag{22}$$

The factors in the product are the Clebsch–Gordan coefficients which are computed to [1]

$$\left| \begin{matrix} j & \frac{1}{2} & j + \frac{1}{2} \\ m & \alpha & m + \alpha \end{matrix} \right|_q = q^{-\alpha j + \frac{m}{2}} \left(\frac{[j + 2\alpha m + 1]}{[2j + 1]} \right)^{\frac{1}{2}} \tag{23}$$

and

$$\left| \begin{matrix} j & \frac{1}{2} & j - \frac{1}{2} \\ m & \alpha & m + \alpha \end{matrix} \right|_q = 2\alpha q^{\alpha(j+1) + \frac{m}{2}} \left(\frac{[j - 2\alpha m]}{[2j + 1]} \right)^{\frac{1}{2}}. \tag{24}$$

As long as q is not a root of unity the above basis is well defined. If r is an integer and $3 \leqslant r \leqslant N$ one has to constrain the set of allowed paths to the restricted Bratelli diagram

$$\Gamma^{(r)} := \{\boldsymbol{j} \in \Gamma | 2j_k + 1 < r, k = 1, \ldots, N\}. \tag{25}$$

Equipped with this particular basis we are in the position to state the first expression for the quasi-Hermiticity operator η.

Theorem 1. *Let* $|\boldsymbol{j}, m\rangle_T$ *denote the complex conjugate path basis, i.e. we set*

$$|\boldsymbol{j}, m\rangle_T := T\,|\boldsymbol{j}, m\rangle = \sum_{|\alpha|=m} \overline{\langle\alpha|\boldsymbol{j}, m\rangle} v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N}, \qquad \boldsymbol{j} \in \Gamma. \tag{26}$$

Then the following sum over projectors,

$$\eta = \sum_{\boldsymbol{j},m} |\boldsymbol{j}, m\rangle_T\ {}_T\langle \boldsymbol{j}, m|\,, \tag{27}$$

defines a positive definite, Hermitian and invertible operator which satisfies (12). Here the sum over j *in (27) is restricted to* $\Gamma^{(r)}$ *for* $3 \leqslant r \leqslant N$ *integer and unrestricted for* $r \in (N, \infty)$.

Henceforth, it shall always be understood that we take the restricted path set $\Gamma^{(r)}$ for the regime (i) and the unrestricted one, Γ, for regime (ii).

Expression 2: η *in terms of quantum group generators*. Given expression (27) it is natural to ask whether the sum over the path states can be performed. The answer is positive.

Theorem 2. *Let* $\eta = RC'$ *with* R *the spin-reversal operator. For fixed* $0 \leqslant j \leqslant N/2$ *denote by* Γ_j *(resp.* $\Gamma_j^{(r)}$*) the subspace spanned by all path vectors with endpoint* $j_N = j$ *(i.e. the direct sum of all spin-*j *modules* V_j *obtained from the Clebsch–Gordan decomposition of* $V^{\otimes N}$ *as discussed above). Then the restriction* C'_j *of the operator* C' *on this subspace can be expressed in terms of the quantum group generators as*

$$C'_j = (-)^{\frac{N}{2}-j} \sum_{m \in \frac{1}{2}\mathbb{N}} \frac{[j-m]_q!}{[j+m]_q!} \frac{(S^-)^{2m}\delta_{S^z,m} + (S^+)^{2m}\delta_{S^z,-m}}{2^{\delta_{0,m}}}. \tag{28}$$

On the path basis this operator acts as follows

$$C'|\boldsymbol{j}, m\rangle = (-)^{\frac{N}{2}-j_N}|\boldsymbol{j}, -m\rangle, \tag{29}$$

where j_N *is the endpoint of the path* $\boldsymbol{j}$. *Thus, we have in particular that* $C'^2 = 1$ *or equivalently*

$$R\eta R = \eta^{-1}. \tag{30}$$

Expression 3: η *in terms of the Hecke algebra*. In order to state the third expression for the quasi-Hermiticity operator we need another algebra first: the Hecke algebra $H_N(q)$ is generated by $N-1$ letters $\{b_i\}_{i=1}^{N-1}$ obeying the defining relations,

$$b_i b_i^{-1} = b_i^{-1} b_i = 1, \qquad b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \qquad b_i b_j = b_j b_i, \quad |i-j| > 1 \tag{31}$$

and the quadratic relation

$$(b_i + q)(b_i - q^{-1}) = 0. \tag{32}$$

Once more we introduce a representation $\pi_H : H_N(q) \to \operatorname{End} V^{\otimes N}$ whose images we denote by capital letters. Using the homomorphism$\varphi : H_N(q) \to TL_N(q)$ with $b_i \mapsto q^{-1} + e_i$ and $b_i^{-1} \mapsto q + e_i$ we extend the previously defined representation (11) of the Temperley–Lieb algebra to the Hecke algebra by setting

$$b_i \mapsto \pi_{TL}(\varphi(b_i)) = B_i = q^{-1} + E_i. \tag{33}$$

Theorem 3. *Set* $\eta = PC$ *with* P *being the parity operator. Then the restriction* C_j *of the operator* C *to the subspace* Γ_j *(resp.* $\Gamma_j^{(r)}$*) can be expressed in terms of the Hecke algebra as follows,*

$$C_j = q^{\frac{N(N-4)}{4} + j(j+1)} \mathcal{B}, \tag{34}$$

Table 2. Commutation relations for the operator η and the two C-operators.

Operator	Hamiltonian	Temperley–Lieb	Quantum group
η	$\eta H = H^*\eta$	$\eta E_k = E_k^*\eta$	$\eta S^{\pm} = S^{\pm}_{\mathrm{op}}\eta$
$C = P\eta$	$[C, H] = 0$	$CE_k = E_{N-k}C$	$[C, S^{\pm}] = [C, S^z] = 0$
$C' = R\eta$	$[C', H] = 0$	$[C', E_k] = 0$	$C'S^{\pm} = S^{\mp}C', C'S^z = -S^zC'$

where $\mathcal{B}$ denotes the image of the following special braid β under the representation (33),

$$\beta = \beta_1\beta_2\cdots\beta_{N-1}, \qquad \beta_n = b_n b_{n-1}\cdots b_1. \tag{35}$$

Moreover, we have the identities (11) and

$$[C, C'] = 0, \qquad P\eta P = \eta^{-1}. \tag{36}$$

Summary of commutation relations. The operators detailed in theorems 1–3 obey the commutation relations summarized in table 2.

Remark. The construction of the operators C and C' allows one to avoid the summation over the entire path space in (27) and, thus, is a practical advantage for explicit computations with the new inner product (8). However, both operators still have an implicit dependence on the decomposition of the state space $V^{\otimes N}$ into the spin modules V_{2j} through the j-dependent scalar factors in (28) and (34). This decomposition, which is described through the paths in Γ_j (resp. $\Gamma_j^{(r)}$), is by no means the only possible way to decompose the state space $V^{\otimes N}$ into quantum group modules, but represents a particular choice. Given any other decomposition of $V^{\otimes N}$ (which might not be necessarily a decomposition into direct sums of modules) these scalar factors have to be replaced by non-trivial (non-diagonal) matrices which are not easily computed, even numerically. It is therefore desirable to find an alternative expression which does not rely on the choice of decomposition of the state space into quantum group modules. Such a construction will be presented in the next section.

3. Graphical calculus and GNS construction

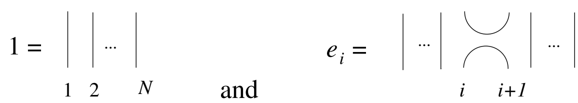
From now on we restrict ourselves to the regime (ii),

$$q = \exp(\mathrm{i}\pi/r), \qquad r \in (N, \infty) \subset \mathbb{R}.$$

This is the regime where the Hamiltonian has no non-trivial Jordan blocks and a reduction of the state space is not necessary. In the first part of this section, we define a family of positive functionals over the Temperley–Lieb algebra in terms of Kauffman diagrams. In the second part, we then employ these functionals to construct self-adjoint representations of $TL_N(q)$ employing the Gel'fand–Naimark–Segal (GNS) construction. In the third part, we state the main conjecture: the unitary equivalence with the self-adjoint representation obtained from the operator η discussed in the previous section.

3.1. Functionals over $TL_N(q)$ in terms of oriented Kauffman diagrams

We start by adopting the well-known graphical representation of the Temperley–Lieb algebra generators e_i in terms of Kauffman diagrams, see the graphical depiction below.



The Temperley–Lieb algebra acts from above by concatenation of the diagrams. We now give these diagrams an orientation by introducing 'arrow configurations' such as

$$\{\underbrace{\uparrow\uparrow\downarrow\downarrow\cdots\uparrow\downarrow\uparrow}_{N}\}.$$

Assigning these arrows (or spins) to the upper and lower ends of the lines in the Kauffman diagrams, we obtain either clockwise, anti-clockwise or unoriented cups and caps. Unoriented lines or arcs are those which join opposing arrows. For instance, the figure below shows two oriented cups (one anti-clockwise, one clockwise) and two unoriented ones.

After these preliminaries we are now ready to state the definition of a family of functionals.

Definition. *For each integer* $0 \leqslant n \leqslant N$ *we define the following arrow configuration (orientation),*

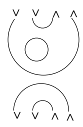
$$s_n = \{\underbrace{\downarrow\downarrow\cdots\downarrow}_{n}\underbrace{\uparrow\uparrow\cdots\uparrow}_{N-n}\}. \tag{37}$$

Denote by $t_a^{(n)}$ *the oriented Kauffman diagram corresponding to* $a \in TL_N(q)$ *with the orientation* s_n. *This diagram will in general contain oriented and unoriented cups, caps and propagating lines as well as closed loops. Denote by* $x_0\big(t_a^{(n)}\big)$ *the number of unoriented lines and arcs (lines and arcs which join a pair of arrows pointing in opposite directions), by* $x\big(t_a^{(n)}\big)$ *the number of anti-clockwise oriented cups (concave arcs only) and by* $y\big(t_a^{(n)}\big)$ *the number of closed loops. Then let* $\omega_n : TL_N(q) \to \mathbb{R}$ *be defined by*

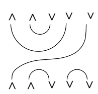
$$a \mapsto \omega_n(a) = \begin{cases} 0, & \text{if} \quad x_0 > 0 \\ (-)^{x+y}(q+q^{-1})^y \frac{q^{\frac{N}{2}-n}+q^{n-\frac{N}{2}}}{q^{\frac{N}{2}-x}+q^{x-\frac{N}{2}}}, & \text{else.} \end{cases} \tag{38}$$

Note that the factor $(-q-q^{-1})^y$ in the definition of ω_n can be omitted if we restrict ourselves to 'reduced' elements of the algebra: using the first relation in (10) all higher powers in the generators can be simplified within the algebra.

Example. In order to illustrate the above definition we present two examples, one for $N = 4$ and for $N = 5$. Choose the number of down spins to be $n = 2$. Then

$N = 4:$ $\qquad \omega_2(e_2^2 e_3 e_1 e_2) =$  $= -(q+q^{-1}),$

since we have $y = 1$, $x_0 = 0$ and $x = 2$. In contrast, one finds for

$N = 5:$ $\qquad \omega_2(e_2 e_1 e_3 e_4 e_2) =$  $= 0$

because there is an unoriented propagating line and an unoriented cap, $x_0 > 0$.

3.2. The GNS construction

First we briefly recall the main facts about the GNS representation. Let $\mathfrak{A}$ be a unital C^*-algebra, i.e. $\mathfrak{A}$ is a complex algebra with unit element 1 and equipped with a conjugation $* : \mathfrak{A} \to \mathfrak{A}$ such that

$$(ab)^* = b^* a^*, (a+b)^* = a^* + b^*, (\lambda a)^* = \bar{\lambda} a^*, a^{**} = a, \qquad a, b \in \mathfrak{A}, \lambda \in \mathbb{C}.$$

Let $\omega : \mathfrak{A} \to \mathbb{C}$ be a 'state' over this algebra, i.e. ω is a positive linear functional of norm 1,

$$\omega(a^* a) \geqslant 0, a \in \mathfrak{A} \qquad \text{and} \qquad \omega(1) = 1. \tag{39}$$

There is a natural action of the algebra on itself (seen as vector space) by assigning to each element $a \in \mathfrak{A}$ the map $\pi(a) : \mathfrak{A} \to \mathfrak{A}$ defined as

$$b \mapsto \pi(a)b = ab, \qquad b \in \mathfrak{A}. \tag{40}$$

Furthermore, we can endow the algebra with an inner product setting

$$\langle a, b\rangle_\omega := \omega(a^* b). \tag{41}$$

In order for this product to be well defined we need to set all elements to zero for which the induced norm vanishes, i.e. we need to take the vector space quotient $\mathfrak{A}/\mathfrak{J}$ with respect to the left ideal

$$\mathfrak{J} = \{a \in \mathfrak{A} \mid \omega(a^* a) = 0\}. \tag{42}$$

Finally, taking the norm completion $\mathfrak{H}^{\mathrm{GNS}} = \overline{\mathfrak{A}/\mathfrak{J}}$ we obtain a Hilbert space and by construction the resulting representation $\pi_\omega : \mathfrak{A} \to \mathrm{End}\, \mathfrak{H}^{\mathrm{GNS}}$ preserves the $*$-structure, i.e. the $*$ -operation in the algebra corresponds to taking the Hermitian adjoint with respect to $\langle \cdot, \cdot \rangle_\omega$. Note also that the representation π_ω is cyclic. That is, there exists a vector Ω (in the present case the equivalence class of$1 \in \mathfrak{A}$) such that $\mathfrak{H}^{\mathrm{GNS}} = \mathfrak{A}\Omega$. We now set $\mathfrak{A} = TL_N(q)$ and fix the $*$-operation on the Temperley–Lieb algebra by requiring that $e_i^* = e_i$. In terms of the corresponding Kauffman diagram this operation corresponds to horizontally flipping the diagram. Then $TL_N(q)$ can be turned into a well-defined C^*-algebra, see e.g. [13]. Following the general outline just given we obtain for each of the above defined functionals ω_n a representation π_n^{GNS} over a Hilbert space $\mathfrak{H}_n^{\mathrm{GNS}}$ with inner product

$$\langle a, b\rangle_{\omega_n} := \omega_n(a^* b). \tag{43}$$

By definition we have that

$$\langle e_i a, b\rangle_{\omega_n} = \omega_n(a^* e_i b) = \langle a, e_i b\rangle_{\omega_n}, \tag{44}$$

whence the GNS representation π_n^{GNS} is self-adjoint, i.e. the inner product is invariant under the $TL_N(q)$ action.

Remark. In order to show that this construction is indeed well defined, one needs to show that the functionals introduced above are normalized and positive. The correct normalization is easily verified, positivity on the other hand is more difficult to show and remains at present a conjecture. However, it has been tested numerically for many examples, $N = 3, 4, \ldots, 8$. Note that positivity would be a direct consequence of the identification with the η-product (8) which we discuss next and which also has been tested numerically.

3.3. The conjecture

Denote by $\mathfrak{H}_n \subset \mathfrak{H} = V^{\otimes N}$ the subspace containing all vectors with n down spins,

$$\mathfrak{H}_n = \operatorname{span}\left\{ v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N} \,\middle|\, \sum \alpha_i = N/2 - n \right\}. \tag{45}$$

As it turns out these spin-sectors viewed as Temperley–Lieb modules with respect to the invariant product $\langle \cdot, \cdot \rangle_\eta$ are in one-to-one correspondence with the GNS modules.

Conjecture. *The self-adjoint representation π_{TL} of the Temperley–Lieb algebra $TL_N(q)$ with $q = \exp(i\pi/r)$, $r > N$ over the Hilbert space $\{\mathfrak{H} = V^{\otimes N}, \langle \cdot, \cdot \rangle_\eta\}$ is unitarily equivalent to the direct sum $\bigoplus_n \pi_n^{\text{GNS}}$ of representations over the Hilbert spaces $\{\mathfrak{H}_n^{\text{GNS}}, \langle \cdot, \cdot \rangle_{\omega_n}\}$. For each fixed spin sector $\mathfrak{H}_n \subset \mathfrak{H} = V^{\otimes N}$ the unitary map*

$$U_n : \mathfrak{H}_n^{\text{GNS}} \to \mathfrak{H}_n \tag{46}$$

is given by

$$a \mapsto U_n a = \pi_{TL}(a)\Omega_n, \qquad \Omega_n = \underbrace{v_{-\frac{1}{2}} \otimes \cdots \otimes v_{-\frac{1}{2}}}_{n} \otimes v_{\frac{1}{2}} \cdots \otimes v_{\frac{1}{2}}. \tag{47}$$

By abuse of notation, we do not distinguish between an algebra element $a \in TL_N(q)$ and its equivalence class with respect to the ideal $\mathfrak{I}_n = \{a | \omega_n(a^* a) = 0\}$. Unitarity means that we have the identity

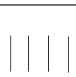
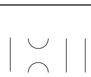
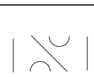
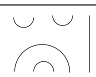
$$\omega_n(a) = \langle \Omega_n, \pi_{TL}(a)\Omega_n \rangle_\eta \qquad \textit{for all} \qquad a \in TL_N(q). \tag{48}$$

Remark. Note that the conjectured identity can also be seen as a *definition* of the inner product $\langle \cdot, \cdot \rangle_\eta$ on $V^{\otimes N}$ via the GNS construction. Namely, for any basis $\{a_i\} \subset TL_N(q)$ in the GNS module $\mathfrak{H}_n^{\text{GNS}}$ we obtain a basis $\{x_i = U_n a_i = \pi_{TL}(a_i)\Omega_n\} \subset \mathfrak{H}_n$ and vice versa. The matrix elements of η with respect to the basis $\{x_i\}$ are then simply obtained from the Gram matrix of the basis $\{a_i\}$ via the relation

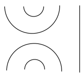
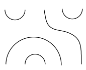
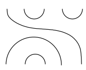
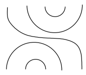
$$\langle x_i, \eta x_j \rangle = \omega_n(a_i^* a_j). \tag{49}$$

This implicit way of introducing η has the advantage of being a basis-independent definition of the new inner product and we have now removed any dependence on the decomposition of the state space with respect to the quantum group action. Moreover, this definition of η provides us with an efficient graphical calculus to compute matrix elements with respect to $\langle \cdot, \cdot \rangle_\eta$. This abstract algebraic reformulation of our original problem might be of help in extending our present results to the wider class of physical systems which have Temperley–Lieb type Hamiltonians.

3.3.1. Example $N = 5$. Let us consider for $N = 5$ the sector $S^z = 1/2, n = 2$. Then the following table gives the correspondence between Kauffman diagrams and basis vectors (algebra elements),

$a \in TL_5$	$a_1 = 1$	$a_2 = e_2$	$a_3 = e_1 e_2$	$a_4 = e_3 e_2$	$a_5 = e_4 e_3 e_2$	$a_6 = e_1 e_3 e_2$
Diagram						

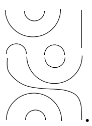
and

$a \in TL_5$	$a_7 = e_2 e_1 e_3 e_2$	$a_8 = e_1 e_4 e_3 e_2$	$a_9 = e_2 e_1 e_4 e_3 e_2$	$a_{10} = e_3 e_2 e_1 e_4 e_3 e_2$
Diagram				

We now state the results for computing the Gram matrix $G = (\omega_2(a_i^* a_j))_{i,j}$ for this particular basis. From the latter one can compute η via (49). Denote by x_{ij} the number of anticlockwise arcs in the diagram associated with $a_i^* a_j$ for $x_0 = 0$ (no unoriented arcs). Likewise, let y_{ij} be the number of closed loops associated with $a_i^* a_j$ for $x_0 = 0$. Both matrices are symmetric, $x_{ij} = x_{ji}$ and $y_{ij} = y_{ji}$. The values of the matrix elements x_{ij}, y_{ij} are given in the table below, whenever $x_0 \neq 0$ we omit the values from the table. Inserting these values into (38) one obtains the Gram matrix G.

x_{ij}/y_{ij}	1	2	3	4	5	6	7	8	9	10
1	0/0	1/0	–	–	–	–	2/0	–	–	–
2		1/1	1/0	1/0	–	2/0	2/1	–	–	–
3			1/1	2/0	–	2/1	2/0	–	–	2/0
4				1/1	1/0	2/1	2/0	2/0	–	–
5					1/1	2/0	–	2/1	–	–
6						2/2	2/1	2/1	2/0	2/1
7							2/2	2/0	2/1	2/0
8								2/2	2/1	2/0
9									2/2	2/1
10										2/2

Again an illustrative example of how to obtain the above values might be helpful. Let us consider the matrix element $G_{6,10} = \omega_2(a_6^* a_{10})$. The Kauffman diagram is obtained by flipping the one for a_6 at the horizontal axis and then connecting it with the one for a_{10} from above,

$$a_6^* a_{10} = e_2 e_3 e_1 a_{10} =$$  .

Obviously, we have $y_{6,10} = 1$. Adding the orientation $\{\downarrow\downarrow\uparrow\uparrow\uparrow\}$ on the top and bottom of the diagram we see that there are no unoriented lines or arcs, $x_0 = 0$, and that we have two anti-clockwise oriented cups, $x_{6,10} = 2$. Hence, $\omega_2(a_6^* a_{10}) = -(q + q^{-1})$.

The reader might wonder why we picked out this particular basis among other choices. The above basis vectors transform particularly simple under the action of the Temperley–Lieb algebra: the only coefficients which occur in the expansion are 0, 1 and powers of $-(q + q^{-1})$. It is in this basis that η takes its simplest form and which is most suitable for the diagonalization of the Hamiltonian. This basis is the analogue of the dual canonical basis introduced in [16] for q real.

4. Conclusions

In this paper, we have presented a conjecture for a new construction of a self-adjoint representation of the Temperley–Lieb algebra $TL_N(q)$ with deformation parameter $q = \exp(\mathrm{i}\pi/r)$, $N < r < \infty$. Clearly, in the thermodynamic limit $N \to \infty$ this section of the unit circle shrinks to the point $q = 1$ where the Hamiltonian is Hermitian and its algebraic structure drastically simplifies.

Nevertheless, this construction is significant as it provides a practical advantage in computing the quasi-Hermiticity operator η for finite lattice size and, as for now, is the simplest starting point for the computation of the square root $\eta^{1/2}$. The latter is needed to obtain the Hamiltonian

$$h = \eta^{1/2} H \eta^{-1/2}, \tag{50}$$

which is Hermitian with respect to the original scalar product on the state space. This Hamiltonian might look very different form the initial Hamiltonian (1) and there is *a priori* no reason to expect that it will have only nearest-neighbour bulk interaction as the similarity transformation is highly nonlocal. Preliminary computations for the case when $q = \exp(\mathrm{i}\pi/2)$, not discussed in [1], show that (50) indeed contains nonlocal bulk interactions. Namely consider the non-Hermitian Hamiltonian

$$H_g = \frac{1}{2}\sum_{m=1}^{M-1}\left[\sigma_m^x\sigma_{m+1}^x + \sigma_m^y\sigma_{m+1}^y + \mathrm{i}g\left(\sigma_m^z - \sigma_{m+1}^z\right)\right], \qquad 0 < g < 1, \tag{51}$$

then one can show for small g its spectral equivalence with the Hermitian Hamiltonian

$$h_{g^2} = -\sum_{n>0}\sum_{x=1}^{M-n} p_x^{(n)}(g^2)\left[c_x^* c_{x+n} - c_x c_{x+n}^*\right], \tag{52}$$

where c_x^*, c_x are fermionic creation and annihilation operators at lattice site x. The hopping probability between a site x and its nth neighbour is encoded in the real coefficients $p_x^{(n)}$. Up to order g^8 one finds that the nonvanishing contributions are

$$\begin{aligned}
p_x^{(1)} &= 1 - \frac{128g^2 + 8g^4 + g^6}{512}(\delta_{x,1} + \delta_{x,M-1}),\\
&\quad - \frac{8g^4 + 3g^6}{512}(\delta_{x,2} + \delta_{x,M-2}) + \frac{g^6}{256}(\delta_{x,3} + \delta_{x,M-3}) + O(g^8),\\
p_x^{(3)} &= \frac{20g^4 + 3g^6}{256}(\delta_{x,1} + \delta_{x,M-3}) + \frac{5g^6}{512}(\delta_{x,2} + \delta_{x,M-4}) + O(g^8)\\
p_x^{(5)} &= -\frac{23g^6}{512}(\delta_{x,1} + \delta_{x,M-5}) + O(g^8),
\end{aligned} \tag{53}$$

A more detailed discussion of this model, which warrants an investigation in its own right, will be presented elsewhere [17].

The close investigation of the regime (ii), $N < r < \infty$, might shed light on how to perform similar GNS constructions for other sections of the unit circle. Such a purely algebraic formulation is desirable in order to apply the results to a wider range of Temperley–Lieb type models, see e.g. [18] and references therein. For instance, by showing the more restrictive identity $\eta E_i = E_i^* \eta$ for the Temperley–Lieb algebra generators the results immediately generalize also to transfer matrices. Namely, it easily follows from the results presented in [1] and this paper that we have constructed a unitary representation of the Hecke algebra

$$\eta B_i = \left(B_i^{-1}\right)^* \eta, \qquad B_i = q^{-1} + E_i. \tag{54}$$

The latter form the basic building blocks for transfer matrices, as for instance the (non-symmetric) double row transfer matrices considered in connection with lattice systems associated with logarithmic minimal models in [19]. With regard to these applications it is natural to ask if the change of the inner product will affect the description in terms of logarithmic conformal field theory when $N \to \infty$ and how this connects to possibly non-local interactions in the bulk. It is planned to address these questions in future work.

Finally, it is worth noting that we discussed in this paper the problem of a non-Hermitian quantum Hamiltonian in the language of C^*-algebras. Instead of using the concepts of quasi-Hermiticity or PT-symmetry, we have considered an associated C^*-algebra of the given quantum-mechanical system and investigated the existence of positive linear functionals. We then used the latter to construct via the GNS approach an inner product with respect to which the Hamiltonian (1) is Hermitian. The formulation of quantum mechanics in the language of C^*-algebras is not new, but the novel aspect in this paper is the presentation of an explicit example where this formulation can be connected with quasi-Hermiticity and PT-symmetry.

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